

# HOPF ACTIONS AND NAKAYAMA AUTOMORPHISMS

K. CHAN, C. WALTON AND J.J. ZHANG

ABSTRACT. Let  $H$  be a Hopf algebra with antipode  $S$ , and let  $A$  be an  $N$ -Koszul Artin-Schelter regular algebra. We study connections between the Nakayama automorphism of  $A$  and  $S^2$  of  $H$  when  $H$  coacts on  $A$  inner-faithfully. Several applications pertaining to Hopf actions on Artin-Schelter regular algebras are given.

## 0. INTRODUCTION

This article is a study in noncommutative invariant theory, particularly on the actions of finite dimensional Hopf algebras on Artin-Schelter regular (AS regular, for short) algebras. This is achieved with the use of Nakayama automorphisms as we explain below.

Let  $k$  be a base field and let  $B$  be either a connected graded AS regular algebra or a noetherian AS regular Hopf algebra. An algebra automorphism  $\mu_B$  of  $B$  is called a *Nakayama automorphism* of  $B$  if there is an integer  $d \geq 0$  such that

$$(E0.0.1) \quad \text{Ext}_{B^e}^i(B, B^e) \cong \begin{cases} \mu_B B^1 & \text{if } i = d, \\ 0 & \text{if } i \neq d \end{cases}$$

as  $B$ -bimodules, where  $B^e = B \otimes B^{op}$  [BZ08, Definition 4.4(b)]. The algebra  $B$  is called *Calabi-Yau* if  $\mu_B = Id$ . Also, the quantity  $d$  is the global dimension of  $B$  when  $B$  is as given above. The definition of  $\mu_B$  is motivated by the classical notion of the Nakayama automorphism of a Frobenius algebra, which will be reviewed in Section 1.

Note that the Nakayama automorphism  $\mu_B$  is unique up to inner automorphism of  $B$ . If  $B$  is connected graded, then the Nakayama automorphism can be chosen to be a graded algebra automorphism, and in this case, it is unique since  $B$  has no non-trivial graded inner automorphism.

Brown and third-named author proved that the Nakayama automorphism of a noetherian AS regular Hopf algebra  $K$  with antipode  $S$  is given by

$$(E0.0.2) \quad \mu_K = S^2 \circ \Xi_{\int^1}^l,$$

---

2000 *Mathematics Subject Classification.* 16E65, 16W30, 16W50, 81R50.

*Key words and phrases.* Artin-Schelter regular algebra, Hopf algebra action.

where  $\Xi_{f^l}^l$  is the left winding automorphism of  $K$  associated to the left homological integral  $\int^l$  of  $K$  [BZ08, Theorem 0.3]. This is an example of how one can express homological invariants (e.g., the Nakayama automorphism) in terms of other invariants (e.g.,  $S^2$  and  $\int^l$ ). Since  $K$  coacts on  $K$  by comultiplication, Equation (E0.0.2) can be interpreted as a linkage between the Nakayama automorphism of the underlying algebra  $K_{\text{alg}}$  of  $K$  and the antipode of the Hopf algebra  $K$  that coacts on  $K_{\text{alg}}$ .

This paper explores connections between properties (as well as numerical invariants) of a connected graded AS regular algebra  $A$  (which is not necessarily a Hopf algebra) and that of a Hopf algebra  $K$ , where  $K$  acts or coacts on  $A$ . One of the key ideas here is to use Manin's construction of quantum linear groups [Man88, Man91] to express  $S^2$  of  $K$  in terms of the Nakayama automorphism of  $A$ .

Before we state our main result, we introduce inner-faithful Hopf actions. Let  $N$  be a right  $K$ -comodule via the comodule structure map  $\rho : N \rightarrow N \otimes K$ . In other words,  $K$  coacts on  $N$ . We say that this coaction is *inner-faithful* if for any proper Hopf subalgebra  $K' \subsetneq K$ , we have that  $\rho(N) \not\subset N \otimes K'$ . A left  $K$ -module  $M$  is called *inner-faithful* if there is no nonzero Hopf ideal  $I \subset K$  such that  $IM = 0$ .

**Theorem 0.1.** *Let  $A$  be a connected graded  $N$ -Koszul AS regular algebra with Nakayama automorphism  $\mu_A$ . Here,  $N \geq 2$ . Let  $K$  be a Hopf algebra with bijective antipode  $S$  coacting on  $A$  from the right. Suppose that the homological codeterminant (Definition 1.5(b)) of the  $K$ -coaction on  $A$  is the element  $D \in K$  and that the  $K$ -coaction on  $A$  is inner-faithful. Then*

$$(E0.1.1) \quad \eta_D \circ S^2 = \eta_{\mu_A^\tau},$$

where  $\eta_D$  is the automorphism of  $K$  defined by conjugating  $D$  and  $\eta_{\mu_A^\tau}$  is the automorphism of  $K$  given by conjugating by the transpose of the corresponding matrix of  $\mu_A$ .

The automorphism  $\eta_D$  of  $K$  is given by  $\eta_D(a) = D^{-1}aD$  for all  $a \in K$ . Moreover, the algebra automorphism  $\eta_{\mu_A^\tau}$  is defined by using “coordinates”, yet it is worth noting that Theorem 0.1 implies that the definition of this automorphism is independent of the choice of coordinates. See Remark 3.4 and the discussion after Lemma 4.1 for more details. We also conjecture that Theorem 0.1 should hold when  $A$  is not necessarily  $N$ -Koszul. To prove this, first we need to resolve a technical issue; see Remark 4.2.

There are some similarities between the equations (E0.0.2) and (E0.1.1), though these two equations are different at first glance. It would be very interesting if both equations come from a single more general equation.

**Question 0.2.** Is there a way of unifying (E0.0.2) and (E0.1.1)?

Theorem 0.1 has several applications to the study of finite dimensional Hopf actions on connected graded AS regular algebras. For the rest of the introduction, we consider Hopf algebra actions (instead of coactions). Further, we impose the following hypotheses for the rest of the article unless stated otherwise.

**Hypothesis 0.3.** *We assume that*

- (i)  *$H$  is a finite dimensional Hopf algebra;*
- (ii)  *$A$  is a connected graded AS regular algebra;*
- (iii)  *$H$  acts on  $A$  inner-faithfully;*
- (iv) *the  $H$ -action on  $A$  preserves the grading of  $A$ .*

The first consequence of Theorem 0.1 is the following result.

**Theorem 0.4.** *Let  $k$  be algebraically closed. Let  $H$  act on a skew polynomial ring  $A = k_p[x_1, \dots, x_n]$  where  $p$  is not a root of unity, then  $H$  is a group algebra.*

A version of Theorem 0.4 holds for multi-parameter skew polynomial rings; see Theorem 4.3.

Since  $H$  ends up being a group algebra in the theorem above, we consider this Hopf action to be *trivial*, to say,  $H$  is either commutative or cocommutative. On the other hand, there are many non-trivial finite dimensional Hopf algebra actions on skew polynomial rings  $k_p[x_1, \dots, x_n]$  where  $p$  is a root of unity. For instance, some interesting non-trivial Hopf algebra actions on  $k_p[x_1, x_2]$  are given in [CKWZ]. Thus, Theorem 0.4 prompts the question below.

**Question 0.5.** Suppose that  $H$  acts on  $A$  under the assumptions of Hypothesis 0.3. If  $A$  does not contain the commutative polynomial ring  $k[x_1, x_2]$  as a subalgebra, is then  $H$  a group algebra?

Theorem 5.10 provides positive evidence for Question 0.5. Another consequence of Theorem 0.1 is following result.

**Theorem 0.6.** *Suppose that  $\text{char } k = 0$  and that  $A$  is  $N$ -Koszul and so-called  $r$ -Nakayama. If the  $H$ -action on  $A$  has trivial homological determinant, then  $H$  is semisimple.*

As a consequence of [KKZ09, Theorem 0.1], if a pair  $(H, A)$  satisfies the hypotheses of Theorem 0.6, then the invariant subring  $A^H$  is AS Gorenstein. Examples of  $r$ -Nakayama algebras include the Sklyanin algebras of dimension 3 and 4, commutative polynomial rings  $k[x_1, \dots, x_n]$ , and the skew polynomial ring  $k_{-1}[x_1, \dots, x_n]$ . Hence, Theorem 0.6 applies to such algebras.

In the case when  $H$  is semisimple, we show that there are no non-trivial Hopf algebra actions on commutative polynomial ring of two variables; compare this to Theorem 0.4.

**Proposition 0.7.** *Suppose  $H$  is semisimple and acts on the commutative polynomial ring  $A = k[x_1, x_2]$ , where  $k$  is an algebraically closed field of characteristic 0. Then  $H$  is a group algebra.*

On the contrary, there exist non-semisimple Hopf algebras acting on  $k[x_1, x_2]$  inner-faithfully [CKWZ]. Moreover, Proposition 0.7 suggests the following question.

**Question 0.8.** Suppose  $H$  is semisimple and acts on a commutative domain over an algebraically closed field  $k$  of characteristic 0 (not necessarily AS regular). Must  $H$  be a group algebra?

In summary, we see that if  $H$  acts on  $A$  inner-faithfully, then algebraic properties of  $A$  affect the structure of  $H$ . Conversely, we conjecture that the algebraic properties of  $H$  should affect the structure of  $A$ .

This article is organized as follows. In Section 1, we provide background material on AS regularity, on Frobenius algebras and classical Nakayama automorphisms, on Hopf algebra actions and the homological (co)determinant for such actions, and on  $r$ -Nakayama algebras. Section 2 introduces several quantum groups associated to a graded algebra; these first appeared in [Man88]. We provide preliminary results on Hopf actions on Frobenius algebras in Section 3. Moreover, we prove the main results, namely Theorems 0.1, 0.4, and 0.6, in Section 4. We also discuss  $N$ -Koszul algebras in Section 4. Finally in Section 5, we provide examples of the main theorems using AS regular algebras of global dimension 2. We also prove Proposition 0.7 in Section 5.

All vector spaces, algebras and rings are over the base field  $k$ . The unmarked tensor  $\otimes$  means  $\otimes_k$ . The  $k$ -linear dual of a vector space  $V$  is denoted by  $V^*$ . In general, we use  $H$  (respectively,  $K$ ) to denote a Hopf algebra that acts (respectively, coacts) on some algebra  $A$ .

## 1. PRELIMINARIES

This section contains some preliminary material that is needed for this article. We refer to [Mon93] for basic definitions regarding Hopf algebras.

**1.1. Artin-Schelter regularity.** This article focuses on the actions of Hopf algebras on certain algebras; such algebras are given as follows.

**Definition 1.1.** Let  $A$  be a connected graded algebra. We say that  $A$  is *Artin-Schelter Gorenstein* (or *AS Gorenstein*) if

- (i)  $A$  has finite injective dimension  $d < \infty$ , and
- (ii)  $\text{Ext}_A^i({}_A k, {}_A A) \cong \text{Ext}_A^i(k_A, A_A) = \begin{cases} 0 & i \neq d \\ k(l) & i = d \end{cases}$  for some  $l \in \mathbb{Z}$ .

We call  $l$  the *AS index* of  $A$ .

If further

(iii)  $A$  has finite global dimension,

then  $A$  is called *Artin-Schelter regular* (or *AS regular*).

In this paper, we are not assuming that  $A$  has finite Gelfand-Kirillov dimension as in the standard definition of AS regularity.

**1.2. Frobenius algebras and classical Nakayama automorphisms.** Next we recall the definition of a Frobenius algebra. Let  $(G, +)$  be an abelian group (such as  $\mathbb{Z}^d$  for  $d \geq 0$ ). A  $G$ -graded, finite dimensional, unital, associative algebra  $B$  is called *Frobenius* if there is a nondegenerate associative bilinear form

$$\langle -, - \rangle : B \times B \rightarrow k$$

which is graded of degree  $-l$  for some  $l \in G$ . If  $G$  is  $\{0\}$ , then this is the classical definition of a Frobenius algebra.

If  $B$  is a connected graded algebra, then  $B$  is a Frobenius algebra if and only if  $B$  is AS Gorenstein of injective dimension zero. We call  $l$  the *AS index* of  $B$ , which agrees with the AS index defined in Definition 1.1(ii) when  $B$  is connected graded. Moreover,  $B$  is Frobenius if the  $k$ -linear dual  $B^*$  is isomorphic to  $B[-l]$  as graded left  $B$ -modules. A nice discussion about Frobenius algebras can be found in [Smi96].

The *classical Nakayama automorphism* of a Frobenius algebra  $B$  is a  $G$ -graded algebra automorphism  $\mu_B$  of  $B$  satisfying

$$\langle a, b \rangle = \langle b, \mu_B(a) \rangle$$

for all  $a, b \in B$ . It is well-known that the  $k$ -linear dual  $B^*$  is isomorphic to  ${}^{\mu_B}B^1(-l)$  as  $B$ -bimodules.

**1.3. Hopf algebra actions.** If  $H$  is a Hopf algebra, we denote by  $H^\circ$  its Hopf dual as in [Mon93, Theorem 9.1.3]. If  $H$  is finite dimensional, then  $H^\circ = H^*$  as a  $k$ -vector space. We say that  $H$  *(co)acts* on an algebra  $A$  if  $A$  arises as an  $H$ -(co)module algebra; refer to [Mon93, Definitions 4.1.1 and 4.1.2] for the definition of a  $H$ -module algebra and a  $H$ -comodule algebra. Note that if  $H$  is finite dimensional, then  $M$  is a left  $H$ -module if and only if  $M$  is a right  $H^\circ$ -comodule. We also provide the definition of inner-faithfulness.

**Definition 1.2.** Let  $M$  be a left  $H$ -module. We say that the  $H$ -action on  $M$  is *inner-faithful* [BB10, Definition 2.7] if  $IM \neq 0$  for any nonzero Hopf ideal  $I \subset H$ . Let  $N$  be a right  $H$ -comodule with comodule structure map  $\rho : N \rightarrow N \otimes H$ . We say that this coaction is *inner-faithful* if  $\rho(N) \not\subset N \otimes H'$  for any proper Hopf subalgebra  $H' \subsetneq H$ .

Here is an easy lemma about inner-faithfulness and its proof is omitted.

**Lemma 1.3.** *Let  $H$  be a Hopf algebra and  $K := H^\circ$ . Let  $M$  be a left  $H$ -module, or equivalently, a right  $K$ -comodule with comodule structure map  $\rho : M \rightarrow M \otimes K$ .*

- (a) *If  $H$  is finite dimensional, then the left  $H$ -action on  $M$  is inner-faithful if and only if the right  $K$ -coaction on  $M$  is inner-faithful.*
- (b) *Suppose  $\{x_i\}$  is a basis of  $M$  and assume that  $\rho(x_i) = \sum_s x_s \otimes a_{si}$  for some elements  $a_{si} \in K$ . Then, the right  $K$ -coaction on  $M$  is inner-faithful if and only if the subalgebra generated by  $\{a_{si}, S^n(a_{si})\}_{s,i,n}$  is  $K$ .*
- (c) *Let  $R$  be an algebra generated by the  $H$ -module  $M$ . Suppose that  $R$  is an  $H$ -module algebra with  $H$ -action induced by the  $H$ -action on  $M$ . Then, the  $H$ -action on  $R$  is inner-faithful if and only if the  $H$ -action on  $M$  is inner-faithful.*  $\square$

**1.4. Homological (co)determinant of a Hopf (co)action.** We now recall the definition of homological determinant in two different settings:

- (1) The original definition given in [KKZ09] when the Hopf algebra  $H$  is finite dimensional and  $A$  is noetherian connected graded AS Gorenstein;
- (2)  $H$  is infinite dimensional and  $A$  is (not necessarily noetherian) AS regular.

Since there is no known uniform definition to cover these cases, we present below the various definitions of homological (co)determinant.

Case (1): Let  $A$  be a noetherian connected graded AS Gorenstein algebra with AS index  $l$ . Let  $d$  be the injective dimension of  $A$  and let  $R^d \Gamma_m(A)$  be the  $d$ -th local cohomology of  $A$ . Then the  $k$ -linear dual  $R^d \Gamma_m(A)^*$  is isomorphic to the graded  $A$ -bimodule  ${}^\mu A^1(-l)$  where  $\mu$  is the Nakayama automorphism of  $A$ . Suppose  $H$  acts on  $A$  such that  $A$  is a left  $H$ -module algebra. When  $H$  is finite dimensional, then  $H$  acts on  $R^d \Gamma_m(A)^*$  from the right; see [KKZ09, Section 3] for more details. Let  $\mathfrak{e} \in R^d \Gamma_m(A)^*$  be a basis element of the lowest degree (i.e., degree  $l$ ). Then, there is an algebra homomorphism  $\gamma : H \rightarrow k$  such that

$$\mathfrak{e} \cdot h = \gamma(h)\mathfrak{e}$$

for all  $h \in H$ .

**Definition 1.4.** Retain the notation above.

- (a) By [KKZ09, Definition 3.3], the composition  $\gamma \circ S : H \rightarrow k$  is called the *homological determinant* of the  $H$ -action on  $A$  and it is denoted by  $\text{hdet}_H A$ . If  $\text{hdet}_H A = \epsilon$ , the counit, then we say that the homological determinant is *trivial*.
- (b) Let  $K(= H^\circ)$  be a Hopf algebra coacting on  $A$  from the right (respectively, from the left) via the  $K$ -comodule map  $\rho : A \rightarrow A \otimes K$  (respectively,  $\rho : A \rightarrow K \otimes A$ ). The *homological codeterminant* of the  $K$ -coaction on  $A$ , denoted by  $\text{hcodet}_K A$ , is defined to be the element  $D \in K$  such that  $\rho(\mathfrak{e}) = \mathfrak{e} \otimes D^{-1}$  (respectively,  $\rho(\mathfrak{e}) = D \otimes \mathfrak{e}$ ). Note that  $D$  and  $D^{-1}$  are

necessarily grouplike (whence invertible) elements. We say the homological codeterminant is *trivial* if  $\text{hcodet}_K A = 1_K$ .

Case (2): Let  $A$  be an AS regular algebra for this case. By [LPWZ08, Corollary D], a connected graded algebra  $A$  is AS regular if and only if the Ext-algebra  $E$  of  $A$  is Frobenius. Here,

$$(E1.5.1) \quad E = \bigoplus_{i \geq 0} \text{Ext}_A^i({}_A k, {}_A k).$$

**Definition 1.5.** Let  $A$  be an AS regular algebra with Frobenius Ext-algebra  $E$  as above. Suppose  $\epsilon$  is a nonzero element in  $\text{Ext}_A^d({}_A k, {}_A k)$  where  $d = \text{gldim } A$ .

- (a) Let  $H$  be a Hopf algebra with bijective antipode  $S$  acting on  $A$  from the left. By [KKZ09, Lemma 5.9],  $H$  acts on  $E$  from the left. The *homological determinant* of the  $H$ -action on  $A$ , denoted by  $\text{hdet}_H A$ , is defined to be  $\eta \circ S$  where  $\eta : H \rightarrow k$  is determined by

$$h \cdot \epsilon = \eta(h)\epsilon.$$

We say the homological determinant is *trivial* if  $\text{hdet}_H A = \epsilon$  where  $\epsilon$  is the counit of  $H$ .

- (b) Let  $K$  be a Hopf algebra with bijective antipode  $S$  coacting on  $A$  from the right (respectively, from the left). By a dual version of [KKZ09, Lemma 5.9], or Remark 1.6(d),  $K$  coacts on  $E$  from the left (respectively, from the right). The *homological codeterminant* of the  $K$ -coaction on  $A$ , denoted by  $\text{hcodet}_K$ , is defined to be  $D$  where  $\rho(\epsilon) = D \otimes \epsilon$  (respectively,  $\rho(\epsilon) = \epsilon \otimes D^{-1}$ ) for some grouplike element  $D \in K$ . We say the homological codeterminant is *trivial* if  $\text{hcodet}_K A = 1_K$ .

Now we make some remarks concerning these definitions. If  $H$  is finite dimensional and if  $A$  is noetherian, then Definition 1.5(a) agrees with [KKZ09, Definition 3.3] by [KKZ09, Lemma 5.10(c)]. Moreover under this assumption, Definition 1.5(b) agrees with [KKZ09, Definition 6.2] by [KKZ09, Remark 6.3 and Lemma 5.10(c)]. Definition 1.5(a) also appears in [LWZ12, Definition 1.11].

Note that the term “homological determinant” is prompted by group actions on commutative polynomial rings. More precisely, in the case that  $A$  is a commutative polynomial ring  $k[x_1, \dots, x_n]$ , and  $H = kG$ , for  $G$  a finite subgroup of  $GL_n(k)$ ,  $\text{hdet}_H A$  becomes the ordinary determinant  $\det_G : G \rightarrow k$  [KKZ09, Remark 3.4(c)]. Hence, the homological determinant of the  $H$ -action  $A$  is trivial in this case if and only if  $G \subseteq SL_n(k)$ .

**Remark 1.6.** Let  $H$  and  $K$  be Hopf algebras. Let  $A$  be a left  $H$ -module algebra, and  $E$  be the Ext-algebra  $\bigoplus_{i \geq 0} \text{Ext}_A^i({}_A k, {}_A k)$ .

- (a) In general,  $E$  is not a left  $H$ -module algebra.
- (b) [KKZ09, Lemma 5.9(d)] If  $H$  has a bijective antipode  $S$ , then the opposite ring  $E^{op}$  of  $E$  is naturally a left  $H$ -module algebra. As a consequence,  $E$  is a right  $H$ -module algebra induced by the left action of  $H$  on  $E^{op}$ .
- (c) The left  $K$ -coaction in Definition 1.5(b) is induced by the right  $K$ -coaction given in [KKZ09, Definition 6.2].
- (d) Suppose  $A$  is AS regular and  $K$  has bijective antipode. If  $A$  is a right  $K$ -comodule algebra, then  $E$  is a left  $K$ -comodule algebra.

*Sketch proof of Remark 1.6(d).* Let  $A - \mathbf{GrMod}^K$  be the category of graded left-right  $(A, K)$ -Hopf modules. It is a Grothendieck category with enough injective objects. Let  $A - \mathbf{Per}^K$  be the full subcategory of  $A - \mathbf{GrMod}^K$  consisting of objects having a finite free resolution. If  $I$  is an injective object in  $A - \mathbf{GrMod}^K$ , then [CG05, Lemma 2.2(3), Propositions 2.8(2) and 2.9] imply that  $\mathrm{Ext}_A^p(M, I) = 0$ , for all  $p > 0$  and for all  $M \in A - \mathbf{Per}^K$ . In other words, every injective object  $I$  in  $A - \mathbf{GrMod}^K$  is  $\mathrm{Hom}_A(M, -)$ -acyclic for any  $M$  in  $A - \mathbf{Per}^K$ . Note that if  $A$  is AS regular, then  $k$  is in  $A - \mathbf{Per}^K$  [SZ97, Proposition 3.1.3]. As a consequence of acyclicity of  $I$ ,  $\mathrm{Ext}_A^i(k, k)$  can be computed by using an injective resolution of the second  $k$  in  $A - \mathbf{GrMod}^K$ . By [CG05, Lemma 2.2(3)], each  $\mathrm{Ext}_A^i(k, k)$  is a right  $K$ -comodule. Therefore  $E$  is a right  $K$ -comodule. Applying the antipode  $S$ , we have that  $E$  is naturally a left  $K$ -comodule. To show that  $E$  is a left  $K$ -comodule algebra, one needs to repeat a similar argument in the proof of [KKZ09, Lemma 5.9], but the details are omitted here.  $\square$

**1.5.  $r$ -Nakayama algebras.** Let  $r \in k^\times$ . Let  $A$  be a connected  $\mathbb{N}$ -graded algebra. Define a graded algebra automorphism  $\xi_r$  of  $A$  by

$$\xi_r(x) = r^{\deg x} x$$

for all homogeneous elements  $x \in A$ .

**Definition 1.7.** Let  $A$  be a connected graded AS Gorenstein algebra such that the Nakayama automorphism exists (defined as in (E0.0.1)) and let  $r \in k^\times$ . We say  $A$  is  $r$ -Nakayama if the Nakayama automorphism of  $A$  is  $\xi_r$ . Note that  $A$  is Calabi-Yau if and only if  $A$  is both 1-Nakayama and AS regular.

In particular, the  $r$ -Nakayama property can be defined for connected graded Frobenius algebras and for AS regular algebras.

## 2. MANIN'S QUANTUM LINEAR GROUPS

In this section, we recall Manin's construction of the quantum groups  $\mathcal{O}_A(M)$ ,  $\mathcal{O}_A(GL)$ ,  $\mathcal{O}_A(SL)$ , and  $\mathcal{O}_A(GL/S^2)$  associated to an algebra  $A$  [Man88]. We review this mainly in the case when  $A$  is a quadratic algebra, but we also remark that the



same ideas apply to non-quadratic algebras at the end of the section. Note that examples of this material are provided in Section 5.

**2.1. Hopf actions on quadratic algebras.** First, we introduce some notation. Let  $A$  be a quadratic algebra generated by  $x_1, x_2, \dots, x_n$  in degree one, subject to the relations

$$r_w := \sum c_w^{ij} x_i x_j = 0$$

for  $w = 1, \dots, m$ , where  $c_w^{ij} \in k$  [Man88, Chapter 5]. We assume that  $\{r_1, \dots, r_m\}$  is linearly independent. We will construct a universal Hopf algebra that coacts on  $A$  (Definition 2.6, Lemma 2.7).

Let  $F$  be a free algebra generated by  $\{y_{ij}\}_{1 \leq i, j \leq n}$ . Consider a bialgebra structure on  $F$  defined by

$$(E2.0.1) \quad \Delta(y_{ij}) = \sum_{s=1}^n y_{is} \otimes y_{sj} \quad \text{and} \quad \epsilon(y_{ij}) = \delta_{ij},$$

for all  $1 \leq i, j \leq n$ . The free algebra  $A' := k\langle x_1, \dots, x_n \rangle$  is a right  $F$ -comodule algebra with comodule structure map  $\rho : A' \rightarrow A' \otimes F$  determined by

$$\rho(x_i) = \sum_{s=1}^n x_s \otimes y_{si}$$

for  $1 \leq i \leq n$ .

**Remark 2.1.** Note that Manin uses left coactions in [Man91, Man88], yet the comodule structure map  $\rho$  above corresponds to a right coaction. By symmetry, all results in [Man88] hold for right coactions. In fact, later in this work, we will use both left and right coactions.

Let  $A = k\langle x_1, \dots, x_n \rangle / (R)$  where  $R = \oplus_{i=1}^m k r_i \subset A_1^{\otimes 2}$ . To determine a coaction of  $K$  on  $A$  induced by the coaction of  $K$  on  $A'$  above, we use the following explicit construction. Let  $V = A_1$  and  $V^*$  be the  $k$ -linear dual of  $V$ . Identify  $(V^*)^{\otimes 2}$  with  $(V^{\otimes 2})^*$  naturally. The *Koszul dual* of  $A$ , denoted by  $A^\dagger$ , is an algebra  $k\langle V^* \rangle / (R^\perp)$  where  $R^\perp$  is the subspace

$$R^\perp := \{a \in (V^*)^{\otimes 2} \mid \langle a, r_w \rangle = 0, \text{ for all } w\} = \{a \in (V^*)^{\otimes 2} \mid \langle a, R \rangle = 0\}.$$

By duality,

$$R = \{r \in V^{\otimes 2} \mid \langle R^\perp, r \rangle = 0\}.$$

Pick a basis for  $R^\perp$ , say  $r'_u$  for  $u = 1, \dots, n^2 - m$ , and write

$$r'_u = \sum_{i,j} d_u^{ij} x_i^* x_j^*$$

for all  $1 \leq u \leq n^2 - m$ . By the definition of  $R^\perp$ , we have that

$$(E2.1.1) \quad \sum_{i,j} d_u^{ij} c_w^{ij} = 0$$

for all  $u, w$ . The following lemma is well-known.

**Lemma 2.2.** [Man88, Lemmas 5.5 and 5.6] *Let  $K$  be a bialgebra coacting on the free algebra  $A' = k\langle x_1, \dots, x_n \rangle$  with  $\rho(x_i) = \sum_{s=1}^n x_s \otimes a_{si}$  for  $a_{ij} \in K$ . Then the following are equivalent.*

- (a) *The coaction  $\rho : A' \rightarrow A' \otimes K$  satisfies  $\rho(R) \subseteq R \otimes K$ , that is,*

$$\sum_{i,j,k,l} c_w^{ij} d_u^{kl} a_{ki} a_{lj} = 0$$

*for all  $w, u$ .*

- (b) *The map  $\rho$  induces naturally a coaction of  $K$  on  $A$  such that  $A$  is a right  $K$ -comodule algebra.  $\square$*

Let  $I$  be the ideal of the bialgebra  $F$ , defined at the beginning of this section, generated by elements  $\sum_{i,j,k,l} c_w^{ij} d_u^{kl} y_{ki} y_{lj}$  for all  $w, u$ . Now we can define the first quantum group associated to  $A$ .

**Definition 2.3.** [Man88, Section 5.3] *The quantum matrix space associated to  $A$  is defined to be  $\mathcal{O}_A(M) = F/I$ , which is a bialgebra quotient of  $F$ .*

It follows that  $\mathcal{O}_A(M)$  is non-trivial since (E2.1.1) gives  $m(n^2 - m)$  quadratic relations between its generators  $y_{ij}$ . Thus,  $\mathcal{O}_A(M)$  is noncommutative in general since for  $\mathcal{O}_A(M)$  to be commutative, we need  $n^2(n^2 - 1)/2$  (quadratic, commutative) relations. Moreover,  $\mathcal{O}_A(M)$  is similar to the quantum group  $\underline{\text{end}}(A)$  defined in [Man88, Section 5.3], but one sees that  $\mathcal{O}_A(M) \cong (\underline{\text{end}}(A))^{\text{coop}}$  because  $\rho$  corresponds to right coaction and left coactions are discussed in [Man88]. On the other hand, we will see in the next proposition that  $\mathcal{O}_A(M)$  coincides with Manin's  $\underline{\text{end}}(A^!)$ .

Before we proceed, we introduce additional notation. Let  $A''$  be the free algebra generated by  $(A_1)^*$ . Define a coaction of  $F$  on  $A''$  via  $\rho^! : A'' \rightarrow F \otimes A''$ , by

$$\rho^!(x_i^*) = \sum_{s=1}^n y_{is} \otimes x_s^*.$$

To say,  $A''$  is a left  $F$ -comodule algebra with the left coaction determined by  $\rho^!$ . We provide more details on  $\rho^!$  and its connection to the homological codeterminant as follows.

**Remark 2.4.** Given a  $K$ -coaction on an AS regular algebra  $A$  with  $K$  having a bijective antipode, we can reinterpret the homological codeterminant via the coaction  $\rho^!$  on the Ext-algebra of  $A$ . Let  $E$  denote the Ext-algebra  $\bigoplus_{i \geq 0} \text{Ext}_A^i(k, k)$ . (If  $A$  is Koszul, then  $E$  is isomorphic to  $A^!$  [Smi96, Theorem 5.9.4].) By Remark 1.6(d),  $E$  is a left  $K$ -comodule algebra and let  $\rho^!$  be the left  $K$ -coaction on  $E$ . Note that  $E$  is Frobenius [LPWZ08, Corollary D]. Let  $\epsilon$  be a nonzero element of maximal degree of  $E$ . Then  $k\epsilon$  is a left  $K$ -comodule. Since  $\text{Ext}_A^1(k, k)$  can be naturally identified with  $(A_1)^*$ , it is not hard to see that the left  $K$ -coaction  $\rho^!$  restricted on  $\text{Ext}_A^1(Ak, {}_A k)$  agrees with the induced left  $K$ -coaction on  $(A_1)^*$ . Now by Definition 1.5(b), the

homological codeterminant of the  $K$ -coaction on  $A$  is defined to be the element  $D \in K$  such that

$$\rho^!(\epsilon) = D \otimes \epsilon.$$

Note that  $D$  is called the *quantum determinant* by Manin [Man88, Section 8.2].

Following [Man88, Section 8.5], we have the following result.

**Proposition 2.5.** [Man88, Theorem 5.10] *Let  $K$  be a bialgebra (not necessarily  $F$  above) coacting on the free algebra  $A' = k\langle x_1, \dots, x_n \rangle$  with  $\rho(x_i) = \sum_{s=1}^n x_s \otimes a_{si}$  for  $a_{ij} \in K$ . Then the following are equivalent.*

- (a) *The coaction  $\rho : A' \longrightarrow A' \otimes K$  satisfies  $\rho(R) \subseteq R \otimes K$ , that is,*

$$\sum_{i,j,k,l} c_w^{ij} d_u^{kl} a_{ki} a_{lj} = 0$$

*for all  $w, u$ .*

- (b) *The map  $\rho$  induces naturally a right coaction of  $K$  on  $A$  such that  $A$  is a right  $K$ -comodule algebra.*  
 (c) *The map  $\rho^!$  induces a natural left coaction of  $K$  on  $A^!$  such that  $A^!$  is a left  $K$ -comodule algebra.*  
 (d) *There is a bialgebra homomorphism  $\phi : \mathcal{O}_A(M) \rightarrow K$  defined by  $\phi(y_{ij}) = a_{ij}$ .*

*Proof.* (a)  $\Leftrightarrow$  (b) This is Lemma 2.2.

- (a)  $\Leftrightarrow$  (c) This follows from a version of Lemma 2.2 for  $(A^!, \rho^!)$ .

- (a)  $\Leftrightarrow$  (d) This follows from the definition of  $\mathcal{O}_A(M)$ . □

For any bialgebra  $B$ , Manin defines the *Hopf envelope* of  $B$  in [Man88, Chapter 7] and we use this to introduce the second quantum group associated to the (quadratic) algebra  $A$ .

**Definition 2.6.** [Man88, page 47] The *quantum general linear group* associated to  $A$  is defined to be the Hopf envelope of  $\mathcal{O}_A(M)$ , and is denoted by  $\mathcal{O}_A(GL)$ . Abusing notation, we use  $y_{ij}$  to denote the image of the generators  $y_{ij}$  of  $\mathcal{O}_A(M)$  in  $\mathcal{O}_A(GL)$ .

We see that  $\mathcal{O}_A(GL)$  serves as the universal Hopf algebra that coacts on  $A$  inner-faithfully.

**Lemma 2.7.** *Let  $A$  be a quadratic algebra generated by  $A_1 = \bigoplus_{i=1}^n kx_i$  and  $K$  be a Hopf algebra coacting on  $A$  from the right. Write  $\rho_K(x_i) = \sum_{s=1}^n x_s \otimes a_{si}$  for some  $a_{si} \in K$ . Then the  $K$ -coaction is inner-faithful if and only if, for all  $1 \leq i, j \leq n$ , the map  $\phi : y_{ij} \rightarrow a_{ij}$  induces a surjective Hopf algebra homomorphism from  $\mathcal{O}_A(GL)$  to  $K$ .*

*Proof.* By Proposition 2.5, there is a bialgebra homomorphism  $\phi : \mathcal{O}_A(M) \rightarrow K$  defined by  $\phi(y_{ij}) = a_{ij}$  for all  $i, j$ . Since  $K$  is a Hopf algebra, by [Man88, Theorem 7.3(c)], this map extends uniquely to a Hopf algebra homomorphism  $\phi : \mathcal{O}_A(GL) \rightarrow K$  where  $\phi(y_{ij}) = a_{ij}$  for all  $i, j$ . Let  $K'$  be the image of  $\phi$ , which is a Hopf subalgebra of  $K$  that coacts on  $A$ . If  $K$ -coaction on  $A$  is inner-faithful, then  $K' = K$ , namely, the map  $\phi$  is surjective. Conversely, if  $K$ -coaction is not inner-faithful, there is a Hopf subalgebra  $K' \subsetneq K$  such that  $\rho(A) \subset A \otimes K'$ . By definition, each  $a_{ij}$  is in  $K'$ . By [Man88, Theorem 7.3(c)],  $\phi$  maps  $\mathcal{O}_A(GL)$  to  $K'$ . Here,  $\phi$  is not surjective.  $\square$

**2.2. Hopf actions on non-quadratic algebras.** In order to describe Hopf actions on  $N$ -Koszul algebras for  $N \geq 3$ , we have to consider connected graded algebras that are not quadratic. The definition of an  $N$ -Koszul algebra will be reviewed in Section 4. For the rest of this section, we consider general connected graded algebras which are not necessarily quadratic and collect some basic facts without proofs. Let  $A$  be connected graded algebra generated by a finite dimensional graded vector space  $W$  and  $K$  be a Hopf algebra coacting on  $A$  via  $\delta : A \rightarrow A \otimes K$  from the right such that:

- (H1) each homogeneous component  $A_i$  is a right  $K$ -comodule;
- (H2)  $A$  is a right  $K$ -comodule algebra; and
- (H3)  $K$ -coaction on  $W$  is inner-faithful.

The following is the first quantum group to which we associate to  $A$ .

**Definition-Lemma 2.8.** [Man88, Section 7.5] *There is a quantum general linear group, denoted by  $\mathcal{O}_A(GL)$  (see Definition 2.6 when  $A$  is quadratic), coacting on  $A$ , via*

$$\rho : A \rightarrow A \otimes \mathcal{O}_A(GL).$$

*This coaction has the universal property: for every  $K$ -coaction  $\delta$  on  $A$ , there is a unique Hopf algebra morphism  $\gamma : \mathcal{O}_A(GL) \rightarrow K$  such that  $\delta = (Id \otimes \gamma) \circ \rho$ .  $\square$*

Further,  $\gamma$  is surjective since the  $K$ -coaction is inner-faithful (see Lemma 2.7). Now to introduce the last two quantum groups associated to  $A$  in this section, suppose that  $A$  is AS regular.

**Definition 2.9.** Let  $A$  be an AS regular algebra. Let  $D$  be the homological code-terminant of the  $\mathcal{O}_A(GL)$ -coaction on  $A$ .

- (a) [Man88, Section 8.5] The *quantum special linear group* associated to  $A$ , denoted by  $\mathcal{O}_A(SL)$ , is defined to be the Hopf algebra quotient  $\mathcal{O}_A(GL)/(D - 1_K)$ .
- (b) Let  $I$  be the Hopf ideal generated by  $\{S^2(a) - a \mid \text{for all } a \in \mathcal{O}_A(GL)\}$ . The *quantum  $S^2$ -trivial linear group* associated to  $A$ , denoted by  $\mathcal{O}_A(GL/S^2)$ , is defined to be the Hopf algebra quotient  $\mathcal{O}_A(GL)/I$ .

Both  $\mathcal{O}_A(SL)$  and  $\mathcal{O}_A(GL/S^2)$  are Hopf algebras coacting on  $A$  from the right such that  $A$  is a comodule algebra. The following lemma follows from the universal property of  $\mathcal{O}_A(GL)$ .

**Lemma 2.10.** *Retain the setting of Definition 2.9 and assume (H1)-(H3).*

- (a) *If the  $K$ -coaction has trivial homological codeterminant, then the quotient morphism  $\gamma : \mathcal{O}_A(GL) \rightarrow K$  factors through  $\mathcal{O}_A(SL)$ .*
- (b) *If  $S^2 = Id_K$ , then the quotient morphism  $\gamma : \mathcal{O}_A(GL) \rightarrow K$  factors through  $\mathcal{O}_A(GL/S^2)$ .*  $\square$

### 3. HOPF ACTIONS ON FROBENIUS ALGEBRAS

Let  $E$  be a connected graded Frobenius algebra. The goal of this section is to study Hopf actions on  $E$ . We first determine the Nakayama automorphism  $\mu_E$  in terms of a basis of  $E_1$ ; see Equation (E3.0.4) below.

Let  $l$  be the highest degree of any nonzero element in  $E$ , and let  $\epsilon$  be a nonzero element in  $E_l$ . Since  $E$  is Frobenius, the multiplication of  $E$  defines a non-degenerate bilinear form

$$E_i \times E_{l-i} \rightarrow k\epsilon \cong k(-l)$$

for each  $0 \leq i \leq l$  [Smi96, Lemma 3.2]. Pick any basis  $\{a_1, \dots, a_n\}$  of  $E_1$ . There is a unique basis of  $E_{l-1}$ , say  $\{b_1, \dots, b_n\}$ , such that

$$(E3.0.1) \quad a_i b_j = \delta_{ij} \epsilon$$

for all  $1 \leq i, j \leq n$ . Moreover, there is a unique basis of  $E_1$ , say  $\{c_1, \dots, c_n\}$ , such that

$$(E3.0.2) \quad b_i c_j = \delta_{ij} \epsilon$$

for all  $1 \leq i, j \leq n$ . Therefore, there is non-singular matrix  $\alpha = (\alpha_{ij})_{i,j=1}^n$  such that

$$(E3.0.3) \quad c_i = \sum_{j=1}^n \alpha_{ij} a_j$$

for all  $i$ . By the definition of Nakayama automorphism,  $\mu_E$  maps  $a_i$  to  $c_i$ , or in other words

$$(E3.0.4) \quad \mu_E(a_i) = \sum_{j=1}^n \alpha_{ij} a_j$$

for all  $i$ . Throughout this work, we refer to  $\alpha = (\alpha_{ij})$  as the matrix associated to  $\mu_E$ .

Now let  $K$  be a Hopf algebra coacting on  $E$  from the left such that  $E$  is a left  $K$ -comodule algebra. The  $K$ -coaction (via comodule structure map  $\rho$ ) preserves the grading of  $E$ . By [Man88, Theorem 5.2.2], there is a canonical map  $\gamma : \mathcal{O}_E(GL) \rightarrow$

$K$  (see Lemma 2.7 when  $E$  is quadratic). So if  $\{y_{ij}\}_{1 \leq i, j \leq n}$  is a set of elements in  $K$  such that

$$(E3.0.5) \quad \rho(a_i) = \sum_{s=1}^n y_{is} \otimes a_s$$

for all  $i$ , then  $\Delta(y_{ij}) = \sum_{s=1}^n y_{is} \otimes y_{sj}$  and  $\epsilon(y_{ij}) = \delta_{ij}$ .

For simplicity, denote the matrix  $(y_{ij})_{n \times n}$  by  $\mathbb{Y}$ . For any matrix  $\mathbb{W} = (w_{ij})_{n \times n}$  in  $M_n(K)$ , write  $S(\mathbb{W}) = (S(w_{ij}))_{n \times n}$ . Choose  $b_i$  and  $c_i$  as in (E3.0.1, E3.0.2) and write

$$\begin{aligned} \rho(b_i) &= \sum_{s=1}^n f_{is} \otimes b_s, \\ \rho(c_i) &= \sum_{s=1}^n g_{is} \otimes c_s \end{aligned}$$

for elements  $f_{is}, g_{is}$  in  $K$ . Note that each  $g_{ij}$  is in the space  $\sum_{s,t} k y_{st}$ . If  $E$  is generated in degree 1, then each  $f_{ij}$  is contained in the subalgebra generated by  $\{y_{st}\}_{s,t}$ . Similarly, let  $\mathbb{F} = (f_{ij})_{n \times n}$  and  $\mathbb{G} = (g_{ij})_{n \times n}$  be the corresponding matrices.

**Lemma 3.1.** *Retain the notation above and let  $\mathbb{I}$  be the  $n \times n$  identity matrix. Let  $\mathbb{D}$  be the homological codeterminant of the left  $K$ -coaction  $\rho$  on  $E$ . (This is applicable to  $E$  in Definition 1.5(b) for instance.) We have the statements below.*

- (a)  $\mathbb{Y}S(\mathbb{Y}) = \mathbb{I} = S(\mathbb{Y})\mathbb{Y}$ .
- (b)  $\mathbb{G}S(\mathbb{G}) = \mathbb{I} = S(\mathbb{G})\mathbb{G}$ .
- (c)  $\mathbb{Y}\mathbb{F}^\tau = \mathbb{D}\mathbb{I}$ . As a consequence,  $S(\mathbb{Y}) = \mathbb{F}^\tau(\mathbb{D}^{-1}\mathbb{I})$ .
- (d)  $S(\mathbb{F})S(\mathbb{Y}^\tau) = \mathbb{D}^{-1}\mathbb{I}$ .
- (e)  $S(\mathbb{G})S(\mathbb{F}^\tau) = \mathbb{D}^{-1}\mathbb{I}$ . As a consequence,  $S(\mathbb{F}^\tau) \cdot \mathbb{D}\mathbb{I} = \mathbb{G}$ .
- (f)  $S^2(\mathbb{Y}) = \mathbb{D}\mathbb{I} \cdot \mathbb{G} \cdot \mathbb{D}^{-1}\mathbb{I}$ .
- (g)  $\mathbb{G} = \alpha\mathbb{Y}\alpha^{-1}$  where  $\alpha = (\alpha_{ij})$ .

*Proof.* (a) By definition  $\rho(a_i) = \sum_{s=1}^n y_{is} \otimes a_s$  for all  $i$ . By the coassociativity of  $\rho$ , we have

$$\Delta(y_{ij}) = \sum_{s=1}^n y_{is} \otimes y_{sj} \quad \text{and} \quad \epsilon(y_{ij}) = \delta_{ij}$$

for all  $i, j$ . The assertion follows from the antipode axiom.

(b) This is similar to (a).

(c) Applying  $\rho$  to the equation  $\delta_{ij}\mathfrak{e} = a_i b_j$ , we have by Definition 1.5(b) that

$$\delta_{ij}\mathbb{D} \otimes \mathfrak{e} = \left( \sum_s y_{is} \otimes a_s \right) \left( \sum_t f_{jt} \otimes b_t \right) = \sum_s y_{is} f_{js} \otimes \mathfrak{e}.$$

This implies that  $\mathbb{Y}\mathbb{F}^\tau = \mathbb{D}\mathbb{I}$ . Since  $\mathbb{D}$  is invertible in  $K$ , we have that  $\mathbb{Y}(\mathbb{F}^\tau \mathbb{D}^{-1}\mathbb{I}) = \mathbb{I}$ . So  $\mathbb{F}^\tau \mathbb{D}^{-1}\mathbb{I}$  is a right inverse of  $\mathbb{Y}$ . By part (a),  $S(\mathbb{Y})$  is an (left and right) inverse of  $\mathbb{Y}$ . Hence,  $S(\mathbb{Y}) = \mathbb{F}^\tau \mathbb{D}^{-1}\mathbb{I}$ .

(d) This follows by applying  $S$  to  $\mathbb{Y}\mathbb{F}^\tau = \mathbb{D}\mathbb{I}$  from part (c), and using the fact  $S$  is an anti-endomorphism of  $K$ . Note that  $\mathbb{D}$  is necessarily grouplike, so  $S(\mathbb{D}) = \mathbb{D}^{-1}$ .

(e) Applying  $\rho$  to the equation  $\delta_{ij}\mathfrak{e} = b_i c_j$ , we have that

$$\delta_{ij}D \otimes \mathfrak{e} = \left( \sum_s f_{is} \otimes b_s \right) \left( \sum_t g_{jt} \otimes c_t \right) = \sum_s f_{is} g_{js} \otimes \mathfrak{e},$$

which implies that  $\mathbb{F}\mathbb{G}^\tau = \mathbb{D}\mathbb{I}$ . Applying  $S$  we obtain that  $S(\mathbb{G})S(\mathbb{F}^\tau) = \mathbb{D}^{-1}\mathbb{I}$ . The last assertion follows by the fact  $S(\mathbb{G})$  has the inverse  $\mathbb{G}$  by part (b).

(f) The assertion follows from parts (c,e).

(g) This follows from applying  $\rho$  to the equations  $c_i = \sum_{j=1}^m \alpha_{ij} a_j$  and linear algebra.  $\square$

We now define an algebra endomorphism  $\eta_{\mu_E}$  of  $K$  dependent on the Nakayama automorphism  $\mu_E$  of the Frobenius algebra  $E$ . First choose a basis  $\{a_i\}_{i=1}^n$  of  $E_1$  and let  $\alpha \in M_n(k)$  be the matrix of  $\mu_E|_{E_1}$  with respect to this basis (c.f. (E3.0.4)). Let  $\mathbb{Y} = (y_{ij})_{n \times n}$  where  $\rho(a_i) = \sum_{s=1}^n y_{is} \otimes a_s$  and define

$$(E3.1.1) \quad (\eta_{\mu_E}(y_{ij}))_{n \times n} = \eta_{\mu_E}(\mathbb{Y}) = \alpha \mathbb{Y} \alpha^{-1}.$$

If the  $K$ -coaction on  $E_1$  is inner-faithful, then the entries of  $\mathbb{Y}$  generate  $K$  as a Hopf algebra. In this case, we show below that (E3.1.1) extends to an algebra endomorphism of  $K$  which is independent of the choice of basis  $\{a_i\}_{i=1}^n$  of  $E_1$ .

**Theorem 3.2.** *Assume that*

- (i)  $E$  is connected graded and Frobenius with Nakayama automorphism  $\mu_E$ ;
- (ii)  $K$  is a Hopf algebra with antipode  $S$  coacting on  $E$  such that the left  $K$ -coaction on  $E_1$  is inner-faithful; and
- (iii)  $\mathbb{D} \in K$  is the homological codeterminant of the left  $K$ -coaction on  $E$ .

Then (E3.1.1) determines a unique Hopf algebra endomorphism of  $K$ , still denoted by  $\eta_{\mu_E}$ , and

$$(E3.2.1) \quad \eta_{\mu_E}(y_{ij}) = \mathbb{D}^{-1} S^2(y_{ij}) \mathbb{D}$$

for all  $i, j$ . As a consequence, the antipode  $S$  of  $K$  is surjective.

*Proof.* By Lemma 3.1(f,g),

$$\mathbb{D}^{-1} S^2(\mathbb{Y}) \mathbb{D} = \mathbb{G} = \alpha \mathbb{Y} \alpha^{-1} = \eta_{\mu_E}(\mathbb{Y})$$

which is (E3.2.1). Let  $\eta_{\mathbb{D}}$  be the conjugation by  $\mathbb{D}$ . Then  $\eta_{\mathbb{D}} \circ S^2$  is a Hopf algebra endomorphism of  $K$ . Equation (E3.1.1) says that  $\eta_{\mu_E} = \eta_{\mathbb{D}} \circ S^2$  when applied to  $y_{ij}$ . Since  $\{y_{ij}\}$  generates  $K$  as a Hopf algebra, there is a unique Hopf algebra endomorphism of  $K$  which extends  $\eta_{\mu_E}$ . We denote this endomorphism by  $\eta_{\mu_E}$  again. By (E3.1.1),  $\eta_{\mu_E} = \eta_{\mathbb{D}} \circ S^2$  on all of  $K$ .

By definition,  $\eta_{\mu_E}$  is bijective when restricted to the space spanned by  $\{y_{ij}\}_{i,j}$ . Therefore,  $\eta_{\mu_E}$  is surjective on  $K$ . Since  $\eta_D$  is an automorphism,  $S^2$  is a surjection. Therefore,  $S$  is a surjection.  $\square$

**Theorem 3.3.** *Suppose that*

- (i)  *$E$  is connected graded and Frobenius;*
- (ii) *the map  $\mu_E \mid E_1$  is scalar multiplication*
- (iii)  *$E$  is a left  $K$ -comodule algebra such that the  $K$ -coaction on  $E_1$  is inner-faithful; and*
- (iv) *the  $K$ -coaction on  $E$  has trivial homological codeterminant.*

*Then  $S^2 = Id$ . If in addition,  $\text{char } k = 0$  (or  $\text{char } k \nmid \dim H$ ), then  $H = K^\circ$  is semisimple.*

*Proof.* By hypothesis (ii) and (E3.1.1),  $\eta_{\mu_E}$  is the identity on  $y_{ij} \in K$  for all  $i, j$  for  $\{y_{ij}\}$  in (E3.0.5). By hypothesis (iii) and Lemma 1.3(b),  $\eta_{\mu_E}$  is the identity on  $K$ . By Theorem 3.2, we have that  $S^2(y) = DyD^{-1}$  for all  $y \in K$ . By hypothesis (iv),  $D = 1$ , so  $S^2 = Id$ . By a theorem of Larson-Radford [LR88, Theorem 2.5], if  $\text{char } k = 0$  (or  $\text{char } k \nmid \dim H$ ), then  $H$  is semisimple.  $\square$

**Remark 3.4.** The automorphism  $\eta_{\mu_E}$  is defined by using a basis of  $\{a_i\}_{i=1}^n$  of  $E_1$ . Theorem 3.2 shows that the construction  $\eta_{\mu_E}$  is independent of the choice of such a basis.

#### 4. PROOF OF THE MAIN RESULTS

In this section, we prove Theorems 0.1, 0.4, and 0.6. First we introduce and provide preliminary results on  $N$ -Koszul algebras. The concept of an  $N$ -Koszul algebra was introduced by Berger in [Ber01]. If  $N = 2$ , then an  $N$ -Koszul algebra is just a usual Koszul algebra. The following is the definition for general  $N$ . Let  $V$  be a finite-dimensional  $k$ -vector space and let  $N$  be an integer larger than 1. Let  $R$  be a subspace of  $V^{\otimes n}$  and let  $A = k\langle V \rangle / (R)$ . The algebra  $A$  is called  $N$ -Koszul if the left trivial  $A$ -module  $k$  has a free resolution of the form

$$\cdots \rightarrow A(-s(i))^{d_i} \rightarrow \cdots \rightarrow A(-s(2))^{d_2} \rightarrow A(-s(1))^{d_1} \rightarrow A \rightarrow k \rightarrow 0$$

where  $s(2j) = Nj$  and  $s(2j+1) = Nj+1$ ; see [Ber01, Definition 2.10] and the discussion in [Ber01, Section 2]. We need the following property about  $N$ -Koszul algebras to prove some of our main results.

**Lemma 4.1.** [BM06, Theorem 6.3] *Let  $A$  be an  $N$ -Koszul AS regular algebra of global dimension  $d$ . Let  $\mu_A$  be the Nakayama automorphism of  $A$  (as defined in the introduction) and let  $\mu_E$  be the Nakayama automorphism of its Ext-algebra  $E$  (as defined in (E1.5.1)). Then  $\mu_E \mid E_1 = (-1)^{d+1}(\mu_A \mid A_1)^*$ .  $\square$*



Let  $A$  be an  $N$ -Koszul AS regular algebra with  $\{x_1, \dots, x_n\}$  a basis of  $A_1$ . Retain the notation in the lemma above, and let  $\mathbb{M}$  be the matrix  $(m_{ij})_{n \times n}$  such that

$$\mu_A(x_i) = \sum_{j=1}^n m_{ij} x_j$$

for all  $i$ . If  $\{x_i^*\}_{i=1}^n$  is the basis of  $E_1 := \text{Ext}_A^1(k, k)$ , which is dual to the basis  $\{x_i\}_{i=1}^n$ , then by Lemma 4.1,

$$\mu_E(x_i^*) = \sum_{j=1}^n (-1)^{d+1} m_{ji} x_j^*$$

for all  $i$ . Note that if we use the notation induced in (E3.0.4), then we have that  $\alpha = (-1)^{d+1} \mathbb{M}^\tau$  is the matrix associated to the Nakayama automorphism of  $E$ .

Now consider the automorphism  $\eta_{\mu_A^\tau}$  on  $K$  defined by conjugating by the transpose of the corresponding matrix  $\mathbb{M}$  of  $\mu_A$ :

$$(E4.1.1) \quad \eta_{\mu_A^\tau}(y_{ij}) = \sum_{s,t=1}^n m_{si} y_{st} n_{jt},$$

for all  $1 \leq i, j \leq n$ . Here,  $(n_{ij})_{n \times n} = \mathbb{M}^{-1}$ .

We are now ready to prove Theorem 0.1.

*Proof of Theorem 0.1.* By [Man88, Section 7.5] (also by Remark 1.6(d)),  $K$  coacts on  $E$  from the left such that  $E$  is a left  $K$ -comodule algebra. If the right  $K$ -coaction on  $A_1$  is given by  $\rho(x_i) = \sum_{j=1}^n x_j \otimes y_{ji}$ , then the left  $K$ -coaction on  $E_1 = \text{Ext}_A^1(k, k)$  is given by  $\rho(x_i^*) = \sum_{j=1}^n y_{ij} \otimes x_i^*$ . Since the  $K$ -coaction on  $A$  is inner-faithful and  $A$  is generated by  $A_1$ , by Lemma 1.3(c), the  $K$ -coaction on  $A_1$  is inner-faithful. Hence the  $K$ -coaction on  $E_1$  is inner-faithful. Therefore by Theorem 3.2,  $\eta_{\mu_E} = \eta_D \circ S^2$ . Here,  $D$  is the homological codeterminant of the left  $K$ -coaction on  $E$ . Also by Definition 1.5(b),  $D$  is the homological codeterminant of the right  $K$ -coaction on  $A$ . By Lemma 4.1, we get that  $\mu_E|_{E_1} = (-1)^{d+1} (\mu_A|_{A_1})^*$ . Now  $\eta_{\mu_E} = \eta_{\mu_A^\tau}$  holds by (E3.1.1), (E4.1.1), and the equation  $\alpha = (-1)^{d+1} \mathbb{M}^\tau$  above. The assertion follows.  $\square$

**Remark 4.2.** We conjecture that a version of Lemma 4.1 holds for general AS regular algebras generated in degree 1. If this were true, then Theorem 0.1 holds for any AS regular algebras that are generated in degree 1, particularly for those that are not necessarily  $N$ -Koszul.

Next we aim to prove Theorem 0.4. Let  $\{p_{ij} \mid i < j\}$  be a set of parameters in  $k^\times$ . For this set, let  $k_{p_{ij}}[x_1, \dots, x_n]$  be the *skew polynomial ring* generated by  $x_1, \dots, x_n$  and subject to the relations

$$x_j x_i = p_{ij} x_i x_j$$

for all  $i < j$ . Here, we take  $p_{ii} = 1$  and  $p_{ji} = p_{ij}^{-1}$  for all  $i < j$ .

**Theorem 4.3.** *Suppose  $k$  is algebraically closed. Let  $A$  be the skew polynomial ring  $k_{p_{ij}}[x_1, \dots, x_n]$ . Suppose an  $l$ -dimensional Hopf algebra  $H$  acts on  $A$  inner-faithfully. Assume that, for each pair  $(i, j)$  of distinct indices,  $\prod_{a=1}^n (p_{ia}p_{aj})$  is not a  $2l$ -th root of unity. Then  $H$  is a group algebra.*

*Proof.* It is known that  $A$  is a Koszul algebra and  $A^!$  is Frobenius. We have a basis  $\{x_1^*, \dots, x_n^*\}$  of  $A_1^!$ , subject to the relations

$$x_j^* x_i^* + p_{ij}^{-1} x_i^* x_j^* \text{ for all } i < j \quad \text{and} \quad (x_i^*)^2 = 0 \text{ for all } i.$$

Let  $\{\hat{x}_1^*, \dots, \hat{x}_n^*\}$  be a basis of  $A_{n-1}^!$  where  $\hat{x}_i^* := x_1^* \cdots x_{i-1}^* x_{i+1}^* \cdots x_n^*$ . Moreover, take  $\mathbf{e}$  to be  $x_1^* \cdots x_n^* \in A_n^!$ . Now  $x_i^* \hat{x}_j^* = \delta_{ij} (-p_{1i}^{-1}) \cdots (-p_{i-1,i}^{-1}) \mathbf{e}$ . Let  $b_j$  denote  $(-p_{1j}) \cdots (-p_{j-1,j}) \hat{x}_j^*$  so that

$$\hat{x}_i b_j = \delta_{ij} \mathbf{e}.$$

Compare this to Equation (E3.0.1).

On the other hand,

$$\begin{aligned} b_i x_j^* &= (-p_{1i}) \cdots (-p_{i-1,i}) \hat{x}_i^* x_j^* \\ &= \delta_{ij} \prod_{a=1}^{i-1} (-p_{ai}) \cdot p_{ii} \cdot \prod_{a=i+1}^n (-p_{ai}) \cdot \mathbf{e}. \end{aligned}$$

Let  $c_j$  denote  $\prod_{a=1}^n (-1)^{n-1} p_{ja} x_j^*$  so that

$$b_i c_j = \delta_{ij} \mathbf{e}.$$

Compare this to Equation (E3.0.2).

By Equations (E3.0.3, E3.0.4), we have that the Nakayama automorphism  $\mu_{A^!}$  is defined by

$$\mu_{A^!}(x_i^*) = (-1)^{n+1} \prod_{a=1}^n p_{ia} x_i^*.$$

Hence the matrix associated to  $\mu_{A^!}$  is the diagonal matrix  $\alpha$  with  $(i, i)$ -entry being  $(-1)^{n+1} \prod_{a=1}^n p_{ia}$ . Now let  $\{y_{ij}\}$  be elements of  $K$  satisfying Equation (E3.0.5) and Remark 2.4, to say

$$\rho^!(x_i^*) = \sum_{s=1}^n y_{is} \otimes x_s^*.$$

Put  $\mathbb{Y} := (y_{ij})$ . Then the  $(i, j)$  entry of  $\alpha \mathbb{Y} \alpha^{-1}$  is  $q_{ij} y_{ij}$ , where  $q_{ij} = \prod_{a=1}^n p_{ia} p_{aj}$ .

Now for  $K = H^\circ$  coacting on  $A$ , we aim to show that  $K$  is commutative. Recall that  $l = \dim K = \dim H < \infty$ . Let  $\eta_D$  be the conjugation automorphism by the homological codeterminant  $D$  of the  $K$ -coaction. Then Theorem 0.1 implies that  $\eta_D \circ S^2$  sends  $y_{ij}$  to  $q_{ij} y_{ij}$ . Since  $S^2(D) = D$ ,  $\eta_D$  commutes with  $S^2$ . Since  $D \in K$ , the order  $o(D)$  of  $D$  divides  $l$  by the Nichols-Zoeller theorem [Mon93, Theorem 3.1.5]. By Radford's theorem, (see [RS02, page 209]), we have that  $o(S^2) \mid 2l$ . So  $m := o(\eta_D \circ S^2)$  divides  $2l$ . For any  $i \neq j$ ,

$$y_{ij} = (\eta_D \circ S^2)^m(y_{ij}) = q_{ij}^m y_{ij}.$$

Since we assume that  $o(q_{ij})$  does not divide  $2l$ , we have that  $q_{ij}^m \neq 1$ . Thus  $y_{ij} = 0$  for all  $i \neq j$ . Thus  $K$  is generated by grouplike elements  $y_{ii}$ . Since  $y_{ij} = 0$  for all  $i \neq j$ ,  $\rho^l(x_i^*) = y_{ii} \otimes x_i^*$  for all  $i$ . Then the relation  $x_j^* x_i^* = -p_{ji} x_i^* x_j^*$  implies that  $y_{ii}$  and  $y_{jj}$  commute. Therefore  $K$  is commutative, and  $H = K^\circ$  is a group algebra as desired [Mon93, Theorem 2.3.1].  $\square$

*Proof of Theorem 0.4.* Here we take  $p_{ij} = p$  for all  $i < j$ . It is direct to check that  $q_{ij} := \prod_{a=1}^n (p_{ia} p_{aj}) = p^{2(j-i)}$  is not a root of unity for all  $i \neq j$ , since  $p$  is not a root of unity. Hence Theorem 0.4 follows from Theorem 4.3.  $\square$

Finally, we prove Theorem 0.6 below.

*Proof of Theorem 0.6.* Retain the notation as above and assume that the hypotheses of Theorem 0.6 hold. When  $A$  is  $r$ -Nakayama of global dimension  $d$ , the matrix  $\mathbb{M}$  corresponding to the Nakayama automorphism  $\mu_A$  is  $(-1)^{d+1} r \mathbb{I}$  by Lemma 4.1. In this case, Theorem 0.1 states that

$$D^{-1} S^2(\mathbb{Y}) D = (\mathbb{M}^T)^{-1} \mathbb{Y} \mathbb{M} = \mathbb{Y}.$$

Since we also assume that the  $K$ -coaction has trivial homological codeterminant, we have  $D = 1$ . Thus

$$S^2(\mathbb{Y}) = \mathbb{Y}$$

or  $S^2$  is the identity on the set  $\{y_{ij}\}$ . By Lemma 1.3(b),  $K$  is generated by  $\{y_{ij}\}$  as a Hopf algebra. Thus  $S^2$  is the identity on  $K$ . Since we assume that  $\text{char } k = 0$ ,  $K$  is semisimple by [LR88, Theorem 2.5].  $\square$

A natural question is when the skew polynomial  $A := k_{p_{ij}}[x_1, \dots, x_n]$  is  $r$ -Nakayama. By the proof of Theorem 0.4 and Lemma 4.1,  $\mu_A(x_i) = (\prod_{a=1}^n p_{ia}) x_i$  for each  $i = 1, \dots, n$ . Hence,  $A$  is  $r$ -Nakayama if and only if  $\prod_{a=1}^n p_{ia} = r$  for all  $i$ . Note that  $\prod_{i=1}^n (\prod_{a=1}^n p_{ia}) = 1$ . This implies that  $r^n = 1$ . So,  $A$  is  $r$ -Nakayama if and only if  $r^n = 1$  and  $p_{1i} = r^{-1} \prod_{a=2}^n p_{ia}$  for every  $i$ . Here,  $\{p_{ij}\}_{2 \leq i < j \leq n}$  are independent variables. A special case occurs when  $p_{ij} = -1$  for all  $1 \leq i < j \leq n$ ; in this case,  $A$  is  $(-1)^{n-1}$ -Nakayama.

## 5. EXAMPLES

In this section, we provide some explicit examples of the main results and of some of the quantum groups discussed in Section 2. In particular, we take  $A$  to be an AS regular algebra of global dimension 2. We show that if such an  $A$  is non-PI, then there are no non-trivial Hopf algebra actions on  $A$  (c.f. Theorem 5.10). We also prove Proposition 0.7 here.

5.1. **For the skew polynomial ring**  $A = A_p := k_p[x_1, x_2]$ . Recall that

$$A_p := k_p[x_1, x_2] = k\langle x_1, x_2 \rangle / (x_2x_1 - px_1x_2).$$

Suppose a Hopf algebra  $K$  coacts on  $A_p$  with

$$\rho(x_i) = x_1 \otimes y_{1i} + x_2 \otimes y_{2i}.$$

for some  $y_{si} \in K$  for  $i = 1, 2$ .

**Lemma 5.1.** *If  $K$  coacts on  $A_p$ , then we get the following relations for  $\{y_{ij}\}$ :*

$$y_{12}y_{11} - py_{11}y_{12} = 0,$$

$$y_{22}y_{21} - py_{21}y_{22} = 0,$$

$$y_{22}y_{11} - py_{21}y_{12} = D,$$

$$y_{11}y_{22} - p^{-1}y_{12}y_{21} = D,$$

where  $D$  is the homological codeterminant of the  $K$ -coaction on  $A_p$ .

*Proof.* Since  $K$  coacts on  $A_p$ , the coaction  $\rho$  maps the relation of  $A_p$  to zero. This means that  $\rho(x_2x_1 - px_1x_2) \in k\langle x_1, x_2 \rangle \otimes K$  generates a 1-dimensional right  $K$ -comodule. Hence, there is a grouplike element  $g \in K$  such that

$$\begin{aligned} \rho(x_2x_1 - px_1x_2) &= (x_2x_1 - px_1x_2) \otimes g \\ &= x_1x_2 \otimes (-pg) + x_2x_1 \otimes g. \end{aligned}$$

By a direct computation, we have that

$$\begin{aligned} \rho(x_2x_1 - px_1x_2) &= x_1^2 \otimes y_{12}y_{11} + x_1x_2 \otimes y_{12}y_{21} + x_2x_1 \otimes y_{22}y_{11} + x_2^2 \otimes y_{22}y_{21} \\ &\quad - p(x_1^2 \otimes y_{11}y_{12} + x_1x_2 \otimes y_{11}y_{22} + x_2x_1 \otimes y_{21}y_{12} + x_2^2 \otimes y_{21}y_{22}) \\ &= x_1^2 \otimes (y_{12}y_{11} - py_{11}y_{12}) + x_1x_2 \otimes (y_{12}y_{21} - py_{11}y_{22}) \\ &\quad + x_2x_1 \otimes (y_{22}y_{11} - py_{21}y_{12}) + x_2^2 \otimes (y_{22}y_{21} - py_{21}y_{22}). \end{aligned}$$

By comparing the coefficients of  $x_i x_j$ 's, we have that

$$y_{12}y_{11} - py_{11}y_{12} = 0,$$

$$y_{22}y_{21} - py_{21}y_{22} = 0,$$

$$y_{22}y_{11} - py_{21}y_{12} = g,$$

$$y_{11}y_{22} - p^{-1}y_{12}y_{21} = g.$$

Using the notation at the beginning at Section 3, we have that

$$A^! = E = \frac{k\langle x_1^*, x_2^* \rangle}{(x_2^*x_1^* + p^{-1}x_1^*x_2^*, (x_1^*)^2, (x_2^*))}.$$

Pick  $\mathbf{e} = x_1^*x_2^* \in E$ . Finally, applying  $\rho^!$  (discussed in Remark 2.4) to  $\mathbf{e}$ , we have that the homological codeterminant  $D$  is given by

$$\rho^!(\mathbf{e}) = D \otimes \mathbf{e} = (y_{11} \otimes x_1^* + y_{12} \otimes x_2^*)(y_{21} \otimes x_1^* + y_{22} \otimes x_2^*).$$

Thus  $D = g = y_{11}y_{22} - p^{-1}y_{12}y_{21}$ . The assertion follows.  $\square$

The following is an application of Lemma 3.1(c).

**Lemma 5.2.** *Retain the above notation.*

$$\begin{aligned} S(y_{11}) &= y_{22}D^{-1}, \\ S(y_{12}) &= -py_{12}D^{-1}, \\ S(y_{21}) &= -p^{-1}y_{21}D^{-1}, \\ S(y_{22}) &= y_{11}D^{-1}. \end{aligned}$$

*Proof.* Let  $a_1 := x_1^*$ ,  $a_2 := x_2^*$  be a basis of  $E_1$ . Now we need a basis  $\{b_1, b_2\}$  of  $E_1$  so that  $a_i b_j = \delta_{ij} \epsilon$ . Here,  $b_1 = x_2^* = a_2$  and  $b_2 = -px_1^* = -pa_1$ . Thus

$$\begin{aligned} \rho^!(b_1) &= y_{21} \otimes x_1^* + y_{22} \otimes x_2^* &= -p^{-1}y_{21} \otimes b_2 + y_{22} \otimes b_1, \text{ and} \\ \rho^!(b_2) &= -py_{11} \otimes x_1^* - py_{12} \otimes x_2^* &= y_{11} \otimes b_2 - py_{12} \otimes b_1. \end{aligned}$$

Therefore, the result follows from Lemma 3.1(c).  $\square$

Towards the proof of Proposition 0.7, let us assume that  $K$  is semisimple. We now get the following relations among  $\{y_{ij}\}$ .

**Lemma 5.3.** *Suppose that  $S^2 = Id$ . Then  $K$  has relations*

$$\begin{aligned} y_{21}y_{11} &= p^{-1}y_{11}y_{21}, \\ y_{22}y_{12} &= p^{-1}y_{12}y_{22}, \\ py_{21}y_{12} &= p^{-1}y_{12}y_{21}, \\ y_{22}y_{11} &= y_{11}y_{22}. \end{aligned}$$

*Proof.* We have from Lemma 3.1(a) that

$$\begin{aligned} y_{22}D^{-1}y_{12} &= py_{12}D^{-1}y_{22}, \\ y_{21}D^{-1}y_{11} &= py_{11}D^{-1}y_{21}, \\ y_{22}D^{-1}y_{11} - py_{12}D^{-1}y_{21} &= 1, \\ y_{11}D^{-1}y_{22} - p^{-1}y_{21}D^{-1}y_{12} &= 1. \end{aligned}$$

Applying  $S^2$  to  $y_{ij}$ , we obtain that

$$\begin{aligned} S^2(y_{11}) &= Dy_{11}D^{-1}, \\ S^2(y_{12}) &= p^2Dy_{12}D^{-1}, \\ S^2(y_{21}) &= p^{-2}Dy_{21}D^{-1}, \\ S^2(y_{22}) &= Dy_{22}D^{-1}. \end{aligned}$$

Since  $S^2 = Id$ , we have that

$$\begin{aligned} Dy_{11} &= y_{11}D, \\ Dy_{12} &= p^{-2}y_{12}D, \\ Dy_{21} &= p^2y_{21}D, \\ Dy_{22} &= y_{22}D. \end{aligned}$$

Hence, the first and last set of equations, along with Lemma 5.1, yield the result.  $\square$

Now we compute the quantum groups  $\mathcal{O}_{A_p}(SL)$  and  $\mathcal{O}_{A_p}(GL/S^2)$  associated to  $A_p$  (Definition 2.9). To do this, we consider a two-parameter family of quantum  $GL_n$ , denoted by  $GL_{\alpha,\beta}(n)$ ; such quantum groups were defined by Takeuchi [Tak90, Section 2]. It follows from the definition that the standard quantum group  $\mathcal{O}_q(GL_n(k))$  is equal to Takeuchi's  $GL_{q,q}(n)$ .

**Proposition 5.4.** *Suppose that a finite dimensional Hopf algebra  $K$  (with antipode  $S$ ) coacts on the skew polynomial ring  $A_p$  with  $S^2 = Id_K$ . Then  $K$  is a Hopf quotient of Takeuchi's quantum group  $GL_{p,p^{-1}}(2)$ . As a consequence, the quantum group  $\mathcal{O}_{A_p}(GL/S^2) = GL_{p,p^{-1}}(2)$ .*

*Proof.* The relations in Lemma 5.1 and 5.3 combined give all relations of quantum group  $GL_{p,p^{-1}}(2)$  as defined in [Tak90, Section 2]. Hence, the assertion follows.  $\square$

**Proposition 5.5.** *Suppose that the  $K$ -coaction on  $A_p$  has trivial homological codeterminant, then  $K$  is a Hopf quotient of  $\mathcal{O}_p(SL_2(k))$ . Thus,  $\mathcal{O}_{A_p}(SL) = \mathcal{O}_p(SL_2(k))$ .*

*Proof.* Since the  $K$ -coaction has trivial cohomological determinant,  $D = 1_K$ . Now the relations in Lemma 5.1, the relations obtained from applying the antipode (computed in Lemma 5.2) to these relations, together with  $D = 1_K$  provide a complete set of relations for  $\mathcal{O}_p(SL_2(k))$ . The assertion follows.  $\square$

Now we are ready to prove Proposition 0.7.

*Proof of Proposition 0.7.* Let  $K = H^\circ$ . Then  $K$  coacts on  $k[x_1, x_2]$  inner-faithfully.

Since  $H$  is semisimple, so is  $K$ . Consequently,  $S_K^2 = Id_K$ . By Proposition 5.4,  $K$  is a Hopf quotient of  $GL_{p,p^{-1}}(2)$ , where  $p = 1$  in this case. Since  $GL_{1,1}(2) = \mathcal{O}(GL_2)$  is commutative, so is  $K$ . Therefore  $H$  is cocommutative. Since  $H$  is finite dimensional over an algebraically closed field,  $H$  is a group algebra as desired.  $\square$

**5.2. For the Jordan plane  $A = A_J := k_J[x_1, x_2]$ .** In this subsection, we provide computations similar to those in the previous subsection for the algebra

$$A_J := k_J[x_1, x_2] = k\langle x_1, x_2 \rangle / (x_2x_1 - x_1x_2 - x_1^2).$$

Suppose that  $A_J$  is a  $K$ -comodule algebra with comodule structure map  $\rho : A_J \rightarrow A_J \otimes K$  defined by

$$\rho(x_i) = x_1 \otimes a_{1i} + x_2 \otimes a_{2i}$$

for some  $a_{si} \in K$  with  $i = 1, 2$ .

**Lemma 5.6.** *Retain the notation above. Let  $D$  be the homological codeterminant of the  $K$ -coaction on  $A_J$ . Then we have that*

$$\begin{aligned} a_{12}a_{11} - a_{11}a_{12} - a_{11}^2 &= -D, \\ a_{12}a_{21} - a_{11}a_{22} - a_{11}a_{21} &= -D, \\ a_{22}a_{11} - a_{21}a_{12} - a_{21}a_{11} &= D, \\ a_{22}a_{21} - a_{21}a_{22} - a_{21}^2 &= 0. \end{aligned}$$

*Proof.* As before in Subsection 5.1, use the coefficients of  $\rho(x_2x_1 - x_1x_2 - x_1^2)$ .  $\square$

The next lemma is an application of Lemma 3.1(c).

**Lemma 5.7.** *Retain the notation above. Then*

$$\begin{aligned} S(a_{11}) &= (a_{22} + a_{21})D^{-1}, \\ S(a_{12}) &= (-a_{11} - a_{12} + a_{21} + a_{22})D^{-1}, \\ S(a_{21}) &= -a_{21}D^{-1}, \\ S(a_{22}) &= (a_{11} - a_{21})D^{-1}. \end{aligned} \quad \square$$

Moreover, the following lemma follows from Lemma 5.7.

**Lemma 5.8.** *Retain the notation above. Then*

$$\begin{aligned} S^2(a_{11}) &= D(a_{11} - 2a_{21})D^{-1}, \\ S^2(a_{12}) &= D(a_{12} + 2a_{11} - 2a_{22} - 4a_{21})D^{-1}, \\ S^2(a_{21}) &= Da_{21}D^{-1}, \\ S^2(a_{22}) &= D(a_{22} + 2a_{21})D^{-1}. \end{aligned} \quad \square$$

Now we will see that  $K$  must be a group algebra as in the skew polynomial case.

**Proposition 5.9.** *Suppose that  $K$  coacts on  $A_J$  inner-faithfully with  $\dim K = l < \infty$ , with  $K$  not necessarily semisimple. If  $\text{char } k = 0$ , then  $K = kG$  where  $G = C_n$  for some  $n$ .*

*Proof.* Using the notation introduced in Section 1, we have that  $K$  is generated by elements  $D, D^{-1}$  and  $\{a_{ij}\}_{i,j=1,2}$ . Let  $\eta_D$  be the conjugation automorphism by  $D$ , namely,  $\eta_D : f \mapsto D^{-1}fD$  for all  $f \in K$ . Lemma 5.8 implies that  $\eta_D \circ S^2$  sends  $a_{11}$  to  $a_{11} - 2a_{21}$ .

On the other hand, since  $S^2(D) = D$ ,  $\eta_D$  commutes with  $S^2$ . Since the subalgebra generated by  $D$  is a Hopf subalgebra of  $K$ , we have that the order  $o(D)$  of  $D$  divides  $l$ . By Radford's theorem [RS02, page 209],  $o(S^2) \mid 2l$ . So  $m := o(\eta_D \circ S^2)$  divides  $2l$ . By a computation,

$$a_{11} = (\eta_D \circ S^2)^m(a_{11}) = a_{11} - 2ma_{21}.$$

Since we assume that  $\text{char } k = 0$ , we have  $2m \neq 0$  in  $k$ . Thus  $a_{21} = 0$ . An argument using  $(\eta_D \circ S^2)^m(a_{12}) = a_{12}$  shows that  $a_{11} - a_{22} = 0$ . Thus  $K$  is generated by two elements  $a_{11}$  and  $a_{12}$  with

$$\Delta(a_{12}) = a_{11} \otimes a_{12} + a_{12} \otimes a_{11}.$$

Then  $a_{11}^{-1}a_{12}$  is a primitive element. Since  $K$  is finite dimensional and  $k = 0$ , there is no non-trivial primitive element. Therefore  $a_{12} = 0$  and  $K$  is generated by  $a_{11}$ . As a consequence,  $K = kC_n$  for some  $n$ .  $\square$

**Theorem 5.10.** *Let  $k$  be algebraically closed and  $A$  be a noetherian non-PI AS regular algebra of dimension two that is generated in degree 1. Suppose a finite dimensional Hopf algebra  $H$  acts on  $A$  inner-faithfully. Then  $H$  is a group algebra.*

*Proof.* Since  $k$  is algebraically closed, every AS regular algebra of global dimension two is isomorphic to either  $k_p[x_1, x_2]$  or  $k_J[x_1, x_2]$ .

Case 1:  $A = k_p[x_1, x_2]$ . Since  $A$  is not PI,  $p$  is not a root of unity. The assertion follows from Theorem 0.4.

Case 2:  $A = k_J[x_1, x_2]$ . Since  $A$  is not PI,  $\text{char } k = 0$ . The assertion follows from Proposition 5.9.  $\square$

Now that we have studied Hopf actions on AS regular algebras of global dimension 2, where the Gelfand-Kirillov dimension of  $H$  is 0. We finish with a question.

**Question 5.11.** Let  $p$  be not a root of unity and  $A = k_p[x_1, x_2]$ . Is there a noncommutative Hopf algebra  $K$  of GKdim 1 coacting on  $A$  inner-faithfully?

**Acknowledgments.** C. Walton and J.J. Zhang were supported by the US National Science Foundation: NSF grants DMS-1102548 and DMS-0855743, respectively.

## REFERENCES

- [BB10] Teodor Banica and Julien Bichon. Hopf images and inner faithful representations. *Glasg. Math. J.*, 52(3):677–703, 2010.
- [Ber01] Roland Berger. Koszulity for nonquadratic algebras. *J. Algebra*, 239(2):705–734, 2001.
- [BM06] Roland Berger and Nicolas Marconnet. Koszul and Gorenstein properties for homogeneous algebras. *Algebr. Represent. Theory*, 9(1):67–97, 2006.
- [BZ08] K. A. Brown and J. J. Zhang. Dualising complexes and twisted Hochschild (co)homology for Noetherian Hopf algebras. *J. Algebra*, 320(5):1814–1850, 2008.
- [CG05] S. Caenepeel and T. Guédénon. On the cohomology of relative Hopf modules. *Comm. Algebra*, 33(11):4011–4034, 2005.
- [CKWZ] K. Chan, E. Kirkman, C. Walton, and J. J. Zhang. Quantum binary polyhedral group and their actions on quantum planes. *in preparation*.
- [KKZ09] E. Kirkman, J. Kuzmanovich, and J. J. Zhang. Gorenstein subrings of invariants under Hopf algebra actions. *J. Algebra*, 322(10):3640–3669, 2009.
- [LPWZ08] Di Ming Lu, John H. Palmieri, Quan Shui Wu, and James J. Zhang. Koszul equivalences in  $A_\infty$ -algebras. *New York J. Math.*, 14:325–378, 2008.



- [LR88] Richard G. Larson and David E. Radford. Finite-dimensional cosemisimple Hopf algebras in characteristic 0 are semisimple. *J. Algebra*, 117(2):267–289, 1988.
- [LWZ12] L. L. Liu, Q. S. Wu, and C. Zhu. Hopf actions on Calabi-Yau algebras. *Contemporary Mathematics*, 562:189–210, 2012.
- [Man88] Yu. I. Manin. *Quantum groups and noncommutative geometry*. Université de Montréal Centre de Recherches Mathématiques, Montreal, QC, 1988.
- [Man91] Yuri I. Manin. *Topics in noncommutative geometry*. M. B. Porter Lectures. Princeton University Press, Princeton, NJ, 1991.
- [Mon93] Susan Montgomery. *Hopf algebras and their actions on rings*, volume 82 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1993.
- [RS02] David E. Radford and Hans-Jürgen Schneider. On the even powers of the antipode of a finite-dimensional Hopf algebra. *J. Algebra*, 251(1):185–212, 2002.
- [Smi96] S. Paul Smith. Some finite-dimensional algebras related to elliptic curves. In *Representation theory of algebras and related topics (Mexico City, 1994)*, volume 19 of *CMS Conf. Proc.*, pages 315–348. Amer. Math. Soc., Providence, RI, 1996.
- [SZ97] Darin R. Stephenson and James J. Zhang. Growth of graded Noetherian rings. *Proc. Amer. Math. Soc.*, 125(6):1593–1605, 1997.
- [Tak90] Mitsuhiro Takeuchi. A two-parameter quantization of  $GL(n)$  (summary). *Proc. Japan Acad. Ser. A Math. Sci.*, 66(5):112–114, 1990.

CHAN: DEPARTMENT OF MATHEMATICS, BOX 354350, UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON 98195, USA

*E-mail address:* `kenhchan@math.washington.edu`

WALTON: DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139, USA

*E-mail address:* `notlaw@math.mit.edu`

ZHANG: DEPARTMENT OF MATHEMATICS, BOX 354350, UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON 98195, USA

*E-mail address:* `zhang@math.washington.edu`